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On annuities under random rates of interest

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Abstract

In the article we consider accumulated values of annuities-certain with yearly payments with independent random interest rates. We focus on general annuities with payments varying in arithmetic and geometric progression which are important basic varying annuities (see Kellison, 1991). They are equivalent to the types studied recently by Zaks (2001). We derive, via recursive relationships, mean and variance formulae of the final values of the annuities. As a consequence, we obtain the moments related to the already discussed cases, which leads to a correction of main results from Zaks (2001).

Subj. Class: IE50;IE51

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1. INTRODUCTION

An annuity is defined as a sequence of payments of a limited duration which we denote by n (see, e.g. Gerber, 1997). The accumulated or final values of annuities are of our interest. Typically, for simplicity, it is assumed that underlying interest rate is fixed and the same for all years. However, the interest rate that will apply in future years is of course neither known nor constant. Thus, it seems reasonable to let interest rates vary in a random way over time, cf. e.g. Kellison (1991).

We assume that annual rates of interest are independent random variables with common mean and variance. We apply this assumption in order to compute, via recursive relationships, fundamental characteristics, namely mean and variance, of the accumulated values of annuities with payments varying in arithmetic and geometric progression (see, e.g. Kellison, 1991). Since these important varying annuities can be reduced to the cases considered by Zaks (2001), we discover several mistakes in main results of Zaks (2001). Of course, *errare humanum est*, but for the benefit of the readers we correct all of them here. Moreover, we note that the errors in main results of Zaks (2001) are 'independent', namely they are not merely a consequence of one incorrect result.

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In Section 2 we introduce basic principles of the theory of annuities. Under the fixed interest assumption we consider accumulated values of standard and non-standard annuities. We analyze different payment scenarios which leads to contradiction to Corollary 2.1 from Zaks (2001). Finally, we introduce the ones with payments varying according to arithmetic and geometric progression. It appears that all important types of annuities (cf. Kellison, 1991) can be obtained as examples of the introduced ones.

In Section 3 we drop the assumption of fixed interest rates and we study the final values of the varying payment annuities under stochastic approach to interest. We consider annual rates of interest to be independent random variables with common mean and variance. Using recursive relations we compute first and second moment as well as variance of the accumulated values. Special cases of the derived results correct Theorem 4.2, Theorem 4.3, Theorem 4.5 and Theorem 4.6 from Zaks (2001).

2. ANNUITIES WITH A FIXED INTEREST RATE

First let us recall basic notation used in the theory of annuities. Suppose that j is the positive annual interest rate and fixed through the period of n years. The annual discount rate d is given by the formula

$$(1 + j)d = j \quad (1)$$

and the annual discount factor v is given by the equation

$$(1 + j)v = 1. \quad (2)$$

Hence we have that

$$v + d = 1. \quad (3)$$

In the article we concentrate on final or accumulated values of annuities. We assume that $k \leq n$ throughout, unless otherwise specified. The accumulated value of an annuity-due with k annual payments of 1 is denoted by $\ddot{s}_{\overline{k}|j}$ and given by the formulae

$$\ddot{s}_{\overline{k}|j} = \frac{(1 + j)^k - 1}{d} \quad (4)$$

and

$$\ddot{s}_{\overline{k}|j} = (1 + j)^k + (1 + j)^{k-1} + \dots + (1 + j) = (1 + j)(1 + \ddot{s}_{\overline{k-1}|j}), \quad (5)$$

where the latter defines the recursive equation for $\ddot{s}_{\overline{k}|j}$.

Let us now consider a standard increasing annuity-due. The accumulated value of such annuity with k annual payments of $1, 2, \dots, k$, respectively is:

$$(I\ddot{s})_{\overline{k}|j} = \frac{\ddot{s}_{\overline{k}|j} - k}{d}. \quad (6)$$

This can be expressed recursively as

$$\begin{aligned} (I\ddot{s})_{\overline{k}|j} &= (1 + j)^k + 2(1 + j)^{k-1} + \dots + k(1 + j) \\ &= (1 + j)(k + (I\ddot{s})_{\overline{k-1}|j}), \end{aligned} \quad (7)$$

which corrects formula (2.8) from Zaks (2001).

The accumulated value of an increasing annuity-due with k annual payments of $1^2, 2^2, \dots, k^2$, respectively is denoted by $(I^2\ddot{s})_{\overline{k}|j}$ and calculated:

$$(I^2\ddot{s})_{\overline{k}|j} = \frac{2(I\ddot{s})_{\overline{k}|j} - \ddot{s}_{\overline{k}|j} - k^2}{d}. \quad (8)$$

The following two equations give the recursive formula for $(I^2\ddot{s})_{\overline{k}|j}$ and set the relationship between $(I^2\ddot{s})_{\overline{k}|j}$ and $\ddot{s}_{\overline{k}|j}$.

$$\begin{aligned}(I^2\ddot{s})_{\overline{k}|j} &= (1+j)^k + 2^2(1+j)^{k-1} + \dots + k^2(1+j) \\ &= (1+j)(k^2 + (I^2\ddot{s})_{\overline{k-1}|j}),\end{aligned}\tag{9}$$

$$(I^2\ddot{s})_{\overline{k}|j} = \frac{(1+v)(\ddot{s}_{\overline{k}|j} + k^2) - 2k - 2k^2}{d^2}.\tag{10}$$

The latter contradicts Corollary 2.1 from Zaks (2001).

In the sequel we will need the following two relations:

$$(I\ddot{s})_{\overline{k-1}|j} = (I\ddot{s})_{\overline{k}|j} - \ddot{s}_{\overline{k}|j},\tag{11}$$

$$(I^2\ddot{s})_{\overline{k-1}|j} = (I^2\ddot{s})_{\overline{k}|j} - 2(I\ddot{s})_{\overline{k}|j} + \ddot{s}_{\overline{k}|j}.\tag{12}$$

Standard decreasing annuities are similar to increasing ones, but the payments are made in the reverse order. The accumulated value of such annuity-due with k annual payments of $n, n-1, \dots, n-k+1$, respectively is denoted by $(D\ddot{s})_{\overline{n,k}|j}$ and given by the formulae:

$$\begin{aligned}(D\ddot{s})_{\overline{n,k}|j} &= n(1+j)^k + (n-1)(1+j)^{k-1} + \dots + (n-k+1)(1+j) \\ &= (1+j)((D\ddot{s})_{\overline{n,k-1}|j} + (n-k+1))\end{aligned}\tag{13}$$

and

$$(D\ddot{s})_{\overline{n,k}|j} = (n+1)\ddot{s}_{\overline{k}|j} - (I\ddot{s})_{\overline{k}|j}.\tag{14}$$

The sum of a standard increasing annuity and its corresponding standard decreasing annuity is of course a constant annuity.

Now let us consider the accumulated value of an annuity-due with payments varying in arithmetic progression (see, e.g. Kellison, 1991). The first payment is p and they subsequently increase by q per period, i.e. they form a sequence $(p, p+q, p+2q, \dots, p+(k-1)q)$. We note that p must be positive but q can be either positive or negative as long as $p+(k-1)q > 0$ in order to avoid negative payments. The accumulated value of such annuity will be denoted by $(\ddot{s}_a)_{\overline{k}|j}^{(p,q)}$ and is defined by

$$(\ddot{s}_a)_{\overline{k}|j}^{(p,q)} = p(1+j)^k + (p+q)(1+j)^{k-1} + \dots + (p+(k-1)q)(1+j).\tag{15}$$

Simple calculations lead to the following relationship.

$$(\ddot{s}_a)_{\overline{k}|j}^{(p,q)} = (p-q)\ddot{s}_{\overline{k}|j} + q(I\ddot{s})_{\overline{k}|j}.\tag{16}$$

Important special cases are the combinations of $p=1$ and $q=0$, $p=1$ and $q=1$, and $p=n$ and $q=-1$.

Example 2.1. If $p=1$ and $q=0$, then $(\ddot{s}_a)_{\overline{k}|j}^{(p,q)}$ becomes the accumulated value of an annuity-due with k annual payments of 1, namely

$$(\ddot{s}_a)_{\overline{k}|j}^{(1,0)} = \ddot{s}_{\overline{k}|j}.\tag{17}$$

Example 2.2. If $p = 1$ and $q = 1$, then $(\ddot{s}_a)_{\overline{k}|j}^{(p,q)}$ becomes the accumulated value of an increasing annuity-due with k annual payments of $1, 2, \dots, k$, respectively, namely

$$(\ddot{s}_a)_{\overline{k}|j}^{(1,1)} = (I\ddot{s})_{\overline{k}|j}. \quad (18)$$

Example 2.3. If $p = n$ and $q = -1$, then $(\ddot{s}_a)_{\overline{k}|j}^{(p,q)}$ becomes the accumulated value of a decreasing annuity-due with k annual payments of $n, n-1, \dots, n-k+1$, respectively, namely

$$(\ddot{s}_a)_{\overline{k}|j}^{(n,-1)} = (D\ddot{s})_{\overline{n,k}|j}. \quad (19)$$

Let us finally consider the accumulated value of an annuity-due with k annual payments varying in geometric progression (see, e.g. Kellison, 1991). The first payment is p and they subsequently increase in geometric progression with common ratio q ($q \neq 1+j$) per period, i.e. they form a sequence $(p, pq^2, pq^3, \dots, pq^{k-1})$. We note that p and q must be positive in order to avoid negative payments. The accumulated value of such annuity will be denoted by $(\ddot{s}_g)_{\overline{k}|j}^{(p,q)}$ and is expressed as

$$\begin{aligned} (\ddot{s}_g)_{\overline{k}|j}^{(p,q)} &= p(1+j)^k + pq(1+j)^{k-1} + pq^2(1+j)^{k-2} + \dots + pq^{k-1}(1+j) \\ &= p(1+j) \frac{(1+j)^k - q^k}{1+j-q}. \end{aligned} \quad (20)$$

Important special cases are the combinations of $p = 1$ and $q = 1$ (cf. Example 2.1), and $p = 1$ and $q = 1+u$, where u ($u \neq j$) denotes a fixed rate of increase of the payments.

Example 2.4. If $p = 1$ and $q = 1$, then $(\ddot{s}_g)_{\overline{k}|j}^{(p,q)}$ becomes the accumulated value of an annuity-due with k annual payments of 1 , namely

$$(\ddot{s}_g)_{\overline{k}|j}^{(1,1)} = \ddot{s}_{\overline{k}|j}. \quad (21)$$

Example 2.5. If $p = 1$ and $q = 1+u$, then $(\ddot{s}_g)_{\overline{k}|j}^{(p,q)}$ becomes the accumulated value of an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \dots, (1+u)^{k-1}$, respectively. Moreover, it easy to see that

$$(\ddot{s}_g)_{\overline{k}|j}^{(1,1+u)} = \frac{(1+j)^k}{1+t} \ddot{s}_{\overline{k}|t}, \quad (22)$$

where t is defined as the solution of

$$1+u = (1+j)(1+t). \quad (23)$$

3. ANNUITIES UNDER RANDOM RATES OF INTEREST

Let us suppose that the annual rate of interest in the k th year is a random variable i_k . We assume that, for each k , we have $E(i_k) = j > 0$ and $Var(i_k) = s^2$, and that i_1, i_2, \dots, i_n are independent random variables. We write

$$E(1+i_k) = 1+j = \mu \quad (24)$$

and

$$E[(1+i_k)^2] = (1+j)^2 + s^2 = 1+f = m, \quad (25)$$

where

$$f = 2j + j^2 + s^2. \quad (26)$$

Obviously

$$\text{Var}(1 + i_k) = m - \mu^2. \quad (27)$$

Next we define r to be the solution of

$$1 + r = \frac{1 + f}{1 + j} \quad (28)$$

and using (26) we have

$$r = j + \frac{s^2}{1 + j}. \quad (29)$$

For a k -year variable annuity-due with annual payments of c_1, c_2, \dots, c_k , respectively, we denote their final value by C_k .

3.1. Payments varying in arithmetic progression. In the case of payments varying in arithmetic progression we have that $c_k = p + (k-1)q$, where $k = 1, 2, \dots, n$.

The final value of an annuity with such payments is given recursively:

$$C_k = (1 + i_k)[C_{k-1} + (p + (k-1)q)] \quad \text{for } k = 2, \dots, n. \quad (30)$$

We can easily find $\mu_k = E(C_k)$ as

$$\begin{aligned} E(C_k) &= E((1 + i_k)[C_{k-1} + (p + (k-1)q)]) \\ &= E(1 + i_k)E(C_{k-1} + (p + (k-1)q)) \end{aligned} \quad (31)$$

from independence of interest rates. Thus we have the recursive equation for $k = 2, \dots, n$:

$$\mu_k = \mu[\mu_{k-1} + (p + (k-1)q)]. \quad (32)$$

We note that $\mu_1 = p(1 + j) = p\mu$. The following lemma stems from (32) and (15).

Lemma 3.1. *If C_k denotes the final value of an annuity-due with annual payments varying in arithmetic progression: $p, p + q, p + 2q, \dots, p + (k-1)q$, respectively and if the annual rate of interest during the k th year is a random variable i_k such that $E(1 + i_k) = 1 + j$ and $\text{Var}(1 + i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then*

$$\mu_k = E(C_k) = (\ddot{s}_a)_{\overline{k}|j}^{(p,q)}. \quad (33)$$

Similarly for the second moment $E(C_k^2)$ we have the recursive equation for $k = 2, \dots, n$:

$$m_k = E(C_k^2) = m[m_{k-1} + 2(p + (k-1)q)\mu_{k-1} + (p + (k-1)q)^2]. \quad (34)$$

We note that $m_1 = p^2m$. In order to compute the second moment we need the following lemma.

Lemma 3.2. *Under the assumptions of Lemma 3.1 we have*

$$m_k = M_{1k} + 2M_{2k}, \quad (35)$$

where

$$M_{1k} = p^2m^k + (p + q)^2m^{k-1} + \dots + (p + (k-1)q)^2m \quad (36)$$

and

$$M_{2k} = (p + q)m^{k-1}\mu_1 + (p + 2q)m^{k-2}\mu_2 + \dots + (p + (k-1)q)m\mu_{k-1}. \quad (37)$$

Proof. We proceed by induction. When $k = 2$, this follows on the equation (34), since $\mu_1 = p(1 + j) = p\mu$ and $m_1 = p^2m$. Assuming our result is true for a given k ($2 \leq k \leq n - 1$), it stems from formula (34) that it is also true for $k + 1$. This concludes the proof by induction. \square

Since, by (24), $1 + f = m$ we can easily find that

$$M_{1k} = p^2\ddot{s}_{\overline{k}|f} + 2pq(I\ddot{s})_{\overline{k-1}|f} + q^2(I^2\ddot{s})_{\overline{k-1}|f}. \quad (38)$$

Now we can apply (11) and (12) in order to derive equivalent expression for M_{1k} .

Lemma 3.3.

$$M_{1k} = (p - q)^2\ddot{s}_{\overline{k}|f} + 2q(p - q)(I\ddot{s})_{\overline{k}|f} + q^2(I^2\ddot{s})_{\overline{k}|f}. \quad (39)$$

Now we shall determine M_{2k} using (16), (37) and the fact that $1 + f = m$. Writing

$$\begin{aligned} M_{2k} &= (p + q)(1 + f)^{k-1}[(p - q)\ddot{s}_{\overline{1}|j} + q(I\ddot{s})_{\overline{1}|j}] \\ &+ (p + 2q)(1 + f)^{k-2}[(p - q)\ddot{s}_{\overline{2}|j} + q(I\ddot{s})_{\overline{2}|j}] + \dots \\ &+ (p + (k - 1)q)(1 + f)[(p - q)\ddot{s}_{\overline{k-1}|j} + q(I\ddot{s})_{\overline{k-1}|j}] \\ &= \frac{d(p - q) + q}{d^2} \left[((p + q)(1 + f)^{k-1}(1 + j) \right. \\ &+ (p + 2q)(1 + f)^{k-2}(1 + j)^2 + \dots + (p + (k - 1)q)(1 + f)(1 + j)^{k-1}) \\ &- ((p + q)(1 + f)^{k-1} + (p + 2q)(1 + f)^{k-2} + \dots \\ &+ (p + (k - 1)q)(1 + f)) \left. \right] - \frac{q}{d} \left[(p + q)(1 + f)^{k-1} \right. \\ &+ 2(p + 2q)(1 + f)^{k-2} + \dots + (k - 1)(p + (k - 1)q)(1 + f) \left. \right] \end{aligned} \quad (40)$$

and applying (28) we obtain the following results.

Lemma 3.4. *Under the assumptions of Lemma 3.1 we have*

$$\begin{aligned} M_{2k} &= \frac{1}{d^2} \left[(p - q)(d(p - q) + q)(1 + j)^k \ddot{s}_{\overline{k}|r} \right. \\ &+ q(d(p - q) + q)(1 + j)^k (I\ddot{s})_{\overline{k}|r} \\ &- (p - q)(d(p - q) + qv) \ddot{s}_{\overline{k}|f} \\ &- q(2d(p - q) + qv)(I\ddot{s})_{\overline{k}|f} - q^2 d(I^2\ddot{s})_{\overline{k}|f} \left. \right]. \end{aligned} \quad (41)$$

Lemma 3.5. *Under the assumptions of Lemma 3.1 we have*

$$\begin{aligned} m_k &= \frac{1}{d^2} \left[(q - p)(d(p - q)(1 + v) + 2qv) \ddot{s}_{\overline{k}|f} \right. \\ &- 2q(d(p - q)(1 + v) + qv)(I\ddot{s})_{\overline{k}|f} \\ &- dq^2(1 + v)(I^2\ddot{s})_{\overline{k}|f} + 2(p - q)(d(p - q) + q)(1 + j)^k \ddot{s}_{\overline{k}|r} \\ &+ 2q(d(p - q) + q)(1 + j)^k (I\ddot{s})_{\overline{k}|r} \left. \right]. \end{aligned} \quad (42)$$

We have thus reached a formula for $E(C_k^2)$. In order to compute $Var(C_k)$ we need yet an expression for $E(C_k)^2$.

Lemma 3.6. *Under the assumptions of Lemma 3.1 we have*

$$\begin{aligned}\mu_k^2 &= \frac{p-q}{d} \left(p - q + \frac{2q}{d} \right) \left(\ddot{s}_{2\bar{k}|j} - 2\ddot{s}_{\bar{k}|j} \right) - \frac{2q(p-q)k}{d} \ddot{s}_{\bar{k}|j} \\ &+ \left(\frac{q}{d} \right)^2 \left((I\ddot{s})_{2\bar{k}|j} - 2(1+kd)(I\ddot{s})_{\bar{k}|j} - k^2 \right).\end{aligned}\quad (43)$$

Proof. It is easy to show that

$$(\ddot{s}_{\bar{k}|j})^2 = \frac{\ddot{s}_{2\bar{k}|j} - 2\ddot{s}_{\bar{k}|j}}{d} \quad (44)$$

and

$$(I\ddot{s})_{\bar{k}|j}^2 = \frac{(I\ddot{s})_{2\bar{k}|j} - 2(1+kd)(I\ddot{s})_{\bar{k}|j} - k^2}{d^2}, \quad (45)$$

cf. Lemma 3.3 and 4.3 from Zaks (2001). From (16) we may write

$$\begin{aligned}\mu_k^2 &= ((p-q)\ddot{s}_{\bar{k}|j} + q(I\ddot{s})_{\bar{k}|j})^2 \\ &= (p-q)^2(\ddot{s}_{\bar{k}|j})^2 + 2q(q-p)\ddot{s}_{\bar{k}|j}(I\ddot{s})_{\bar{k}|j} + q^2(I\ddot{s})_{\bar{k}|j}^2.\end{aligned}\quad (46)$$

Substituting from (44), (45) and (6) completes the proof. \square

Now, we are allowed to state the following theorem.

Theorem 3.1. *Under the assumptions of Lemma 3.1 we have*

$$E(C_k) = (\ddot{s}_a)_{\bar{k}|j}^{(p,q)}, \quad (47)$$

$$Var(C_k) = m_k - \mu_k^2, \quad (48)$$

where m_k is given by Lemma 3.5 and μ_k^2 by Lemma 3.6.

Let us now consider the situation when $p = 1$ and $q = 0$. We know, from Example 2.1, that it is the case of an annuity-due with k annual payments of 1. Then we obtain the following corollary (cf. Theorem 3.2 from Zaks, 2001).

Corollary 3.1. *If C_k denotes the final value of an annuity-due with k annual payments of 1 and if the annual rate of interest during the k th year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(1+i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then*

$$E(C_k) = \ddot{s}_{\bar{k}|j}, \quad (49)$$

$$Var(C_k) = \frac{2(1+j)^{k+1}\ddot{s}_{\bar{k}|r} - (2+j)\ddot{s}_{\bar{k}|f} - (1+j)\ddot{s}_{2\bar{k}|j} + 2(1+j)\ddot{s}_{\bar{k}|j}}{j}. \quad (50)$$

Another important case is the combination of $p = 1$ and $q = 1$, see Example 2.2. It is an annuity-due with k annual payments of $1, 2, \dots, k$. The following corollary is a direct consequence of Lemma 3.1, 3.3, 3.4 and 3.5.

Corollary 3.2. *If C_k denotes the final value of an increasing annuity-due with k annual payments of $1, 2, \dots, k$, respectively and if the annual rate of interest during the k th year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(1+i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then*

- (a) $E(C_k) = (I\ddot{s})_{\bar{k}|j}$,
- (b) $M_{1k} = (I^2\ddot{s})_{\bar{k}|f}$,
- (c) $M_{2k} = \frac{(1+j)^{k+2}(I\ddot{s})_{\bar{k}|r} - (1+j)(I\ddot{s})_{\bar{k}|f} - j(1+j)(I^2\ddot{s})_{\bar{k}|f}}{j^2}$,

$$(d) \quad m_k = \frac{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - 2(1+j)(I\ddot{s})_{\overline{k}|f} - j(2+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2}.$$

Part (b) of Corollary 3.2 corrects (4.6) from Zaks (2001) and (d) is in contradiction to Theorem 4.2 from Zaks (2001). These results can be summarized in the following corollary.

Corollary 3.3. *Under the assumptions of Corollary 3.2 we have*

$$E(C_k) = (I\ddot{s})_{\overline{k}|j}, \quad (51)$$

$$\begin{aligned} \text{Var}(C_k) &= \frac{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - 2(1+j)(I\ddot{s})_{\overline{k}|f} - j(2+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2} \\ &\quad - \frac{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}{d^2}. \end{aligned} \quad (52)$$

Corollary 3.3 (variance part) contradicts Theorem 4.3 from Zaks (2001). Let us finally consider the situation when $p = n$ and $q = -1$, see Example 2.3. Then we obtain the following corollary.

Corollary 3.4. *If C_k denotes the final value of a decreasing annuity-due with k annual payments of $n, n-1, \dots, n-k+1$, respectively and if the annual rate of interest during the k th year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\text{Var}(1+i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then*

$$E(C_k) = (D\ddot{s})_{\overline{n,k}|j}, \quad (53)$$

$$\begin{aligned} \text{Var}(C_k) &= \frac{\ell}{d^2} \left[\frac{(n-1/j)^2(1+j)^{2k}\ddot{s}_{\overline{k}|\ell}}{1+\ell} - \frac{2(n-1/j)^2(1+j)^k\ddot{s}_{\overline{k}|r}}{1+r} \right. \\ &\quad + \frac{(n-1/j)^2\ddot{s}_{\overline{k}|f}}{1+f} + \frac{2(n-1/j)(1+j)^k(I\ddot{s})_{\overline{k}|r}}{1+r} \\ &\quad \left. - \frac{2(n-1/j)(I\ddot{s})_{\overline{k}|f}}{1+f} + \frac{(I^2\ddot{s})_{\overline{k}|f}}{1+f} \right], \end{aligned} \quad (54)$$

where $\ell = (s/(1+j))^2$.

Corollary 3.4 (variance part) corrects Theorem 4.5 from Zaks (2001).

3.2. Payments varying in geometric progression. In the case of annuities-due with payments varying in geometric progression we have that $c_k = pq^{k-1}$, where $k = 1, 2, \dots, n$. We assume that p and q are positive, $q \neq 1+j$, $q^2 \neq 1+f$ and $q \neq 1+r$.

The final value of that annuity is given recursively:

$$C_k = (1+i_k)[C_{k-1} + pq^{k-1}]. \quad (55)$$

Similarly, as in the case of payments varying in arithmetic progression, we easily find that for $k = 2, \dots, n$

$$\mu_k = E(C_k) = \mu[\mu_{k-1} + pq^{k-1}]. \quad (56)$$

The second moment $E(C_k^2)$ is given by

$$m_k = E(C_k^2) = m[m_{k-1} + 2pq^{k-1}\mu_{k-1} + p^2q^{2(k-1)}]. \quad (57)$$

We note that $\mu_1 = p(1+j) = p\mu$ and $m_1 = p^2m$. In analogy with Lemma 3.1 we obtain a pleasing form of $E(C_k)$.

Lemma 3.7. *If C_k denotes the final value of an annuity-due with k annual payments varying in geometric progression: $p, pq, pq^2, \dots, pq^{k-1}$, respectively and if the annual rate of interest during the k th year is a random variable i_k such that $E(1+i_k) = 1+j$ and $Var(1+i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then*

$$\mu_k = E(C_k) = (\ddot{s}_g)_{\overline{k}|j}^{(p,q)}. \quad (58)$$

Similarly, as in the previous point, in order to find a formula for $Var(C_k)$ we are about to compute m_k and μ_k^2 . We commence by calculating m_k .

Lemma 3.8. *Under the assumptions of Lemma 3.7 we have*

$$\begin{aligned} m_k &= p^2 m^k + p^2 q^2 m^{k-1} + \dots + p^2 q^{2(k-1)} m \\ &+ 2[pq m^{k-1} \mu_1 + pq^2 m^{k-2} \mu_2 + \dots + pq^{k-1} m \mu_{k-1}]. \end{aligned} \quad (59)$$

Proof. The thesis follows by induction, applying (57) and the fact $1+f=m$. \square

Let

$$M_{1k} = p^2 m^k + p^2 q^2 m^{k-1} + \dots + p^2 q^{2(k-1)} m \quad (60)$$

and

$$M_{2k} = pq m^{k-1} \mu_1 + pq^2 m^{k-2} \mu_2 + \dots + pq^{k-1} m \mu_{k-1}. \quad (61)$$

Hence

$$m_k = M_{1k} + 2M_{2k} \quad (62)$$

(cf. Lemma 3.2). Since $1+f=m$, we can easily obtain an elegant expression for M_{1k} .

Lemma 3.9.

$$M_{1k} = p^2(1+f) \frac{(1+f)^k - q^{2k}}{1+f-q^2} = (\ddot{s}_g)_{\overline{k}|f}^{(p^2, q^2)}. \quad (63)$$

Now we rewrite (61) applying $1+f=m$ and $1+f=(1+j)(1+r)$ giving

$$\begin{aligned} M_{2k} &= pq(1+f)^{k-1} p(1+j) \frac{(1+j)-q}{1+j-q} \\ &+ pq^2(1+f)^{k-2} p(1+j) \frac{(1+j)^2 - q^2}{1+j-q} + \dots \\ &+ pq^{(k-1)}(1+f) p(1+j) \frac{(1+j)^{k-1} - q^{k-1}}{1+j-q} \\ &= \frac{p^2(1+j)}{1+j-q} \left[(q(1+f)^{k-1}(1+j) + q^2(1+f)^{k-2}(1+j)^2 + \dots \right. \\ &+ q^{k-1}(1+f)(1+j)^{k-1} + (1+f)^k - (1+f)^k) - (q^2(1+f)^{k-1} \\ &+ q^4(1+f)^{k-2} + \dots + q^{2(k-1)}(1+f) + (1+f)^k - (1+f)^k) \left. \right] \\ &= \frac{p^2(1+j)}{1+j-q} \left[(1+j)^k(1+r) \frac{(1+r)^k - q^k}{1+r-q} - \frac{(\ddot{s}_q)_{\overline{k}|f}^{(p^2, q^2)}}{p^2} \right]. \end{aligned} \quad (64)$$

Therefore we may write the following lemma.

Lemma 3.10.

$$M_{2k} = \frac{p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)} - (1+j)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q} \quad (65)$$

By virtue of Lemma 3.9 and 3.10, and the fact that $m_k = M_{1k} + 2M_{2k}$ we have the following lemma.

Lemma 3.11. *Under the assumptions of Lemma 3.7 we have*

$$m_k = \frac{2p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)} - (q+1+j)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q}. \quad (66)$$

Thus, we have reached a formula for $E(C_k^2)$. Now we need to derive an expression for $E(C_k)^2$.

Lemma 3.12. *Under the assumptions of Lemma 3.7 we have*

$$\mu_k^2 = \frac{p(1+j)}{1+j-q} \left((\ddot{s}_g)_{\overline{2k}|j}^{(p,q)} - 2q^k(\ddot{s}_g)_{\overline{k}|j}^{(p,q)} \right). \quad (67)$$

Proof. From Lemma 3.7 and (20), we have that

$$\begin{aligned} \mu_k^2 &= \left(p(1+j) \frac{(1+j)^k - q^k}{1+j-q} \right)^2 \\ &= p^2(1+j)^2 \frac{(1+j)^{2k} - 2(1+j)^k q^k + q^{2k}}{(1+j-q)^2} \\ &= \frac{p^2(1+j)^2}{1+j-q} \left(\frac{(1+j)^{2k} - q^{2k}}{1+j-q} - \frac{2q^k((1+j)^k - q^k)}{1+j-q} \right), \end{aligned} \quad (68)$$

which using (20) completes the proof. \square

Since $\text{Var}(C_k) = m_k - \mu_k^2$, we have following theorem.

Theorem 3.2. *Under the assumptions of Lemma 3.7 we have*

$$\begin{aligned} \text{Var}(C_k) &= \frac{2p(1+j)^{k+1}(\ddot{s}_g)_{\overline{k}|r}^{(p,q)} - (1+j+q)(\ddot{s}_g)_{\overline{k}|f}^{(p^2,q^2)}}{1+j-q} \\ &\quad - \frac{p(1+j) \left((\ddot{s}_g)_{\overline{2k}|j}^{(p,q)} - 2q^k(\ddot{s}_g)_{\overline{k}|j}^{(p,q)} \right)}{1+j-q}. \end{aligned} \quad (69)$$

An important case, see Example 2.4, is the combination of $p = 1$ and $q = 1$. Then we obtain an annuity-due with k annual payments of 1 and Theorem 3.2 yields Corollary 3.1.

Another important case is the combination of $p = 1$ and $q = 1+u$, where u ($u \neq j$) denotes a fixed rate of increase of the payments. This defines an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \dots, (1+u)^{k-1}$, respectively (see Example 2.5). We assume also that $1+u = (1+j)(1+t)$, $1+f = (1+u)^2(1+h)$ and $1+f = (1+j)^2(1+t)(1+w)$. This leads to the following corollary.

Corollary 3.5. *If C_k denotes the final value of an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \dots, (1+u)^{k-1}$, respectively and if the annual rate of*

interest during the k th year is a random variable i_k such that $E(1 + i_k) = 1 + j$ and $Var(1 + i_k) = s^2$, and i_1, i_2, \dots, i_n are independent random variables, then

$$E(C_k) = (\ddot{s}_g)_{\overline{k}|j}^{(1,1+u)} = \frac{(1+j)^k \ddot{s}_{\overline{k}|t}}{1+t},$$

$$Var(C_k) = \frac{(1+u)^{2k}(2+t)\ddot{s}_{\overline{k}|h} - 2(1+j)^{2k}(1+t)^k \ddot{s}_{\overline{k}|w}}{t} - \frac{(1+j)^{2k}(\ddot{s}_{\overline{2k}|t} - 2\ddot{s}_{\overline{k}|t})}{t(1+t)}.$$

Corollary 3.5 (variance part) contradicts Theorem 4.6 from Zaks (2001).

Finally, we note that one can approximate mean and variance of the final values of general annuities applying numerical approach. The procedure is as follows (cf. Kellison, 1991).

- (1) Make an appropriate assumption about the probability function for i_k . This uniquely defines the parameters j and s^2 .
- (2) Using standard simulation techniques compute m sets of values for i_1, i_2, \dots, i_k .
- (3) For each of the m sets i_1, i_2, \dots, i_k compute the required accumulated value.
- (4) The m outcomes are used to compute sample mean and variance.

As a result we obtain an approximation for EC_k and $VarC_k$. We may compare them with analytical results.

In order to apply the procedure let us assume that random variables i_k have common normal distribution with parameters $\mu = 0.08$ and $\sigma = 0.02$. This yields that $j = 0.08$ and $s^2 = 0.0004$. Moreover, we set $n = 10$ and $m = 100\,000$. Since the derived formulae for expected accumulated values agree with the ones from Zaks (2001), we focus on the variance results. We plot $Var(C_k)$ as a function of k for the preceding annuity using the analytical and numerical outcomes.

Figure 1 depicts the comparison for an annuity with payments varying in geometric progression with $p = 1$ and $q = 1 + u$ (see Example 2.5), where we set $u = 0.1$. We can observe in the left panel that our analytical (see Corollary 3.5) and numerical results agree while the corresponding Theorem 4.6 from Zaks (2001) yields outcome which is essentially smaller (right panel). Figure 1 shows that the variance of the final values of annuities in ref. [3] can be 100 000 (!!!) times smaller than the correct number. Moreover, the variance is negative for $k = 1$. Therefore, this tool proves to be useful for the verification of the analytical results.

We conducted similar tests for general annuities with payments varying in arithmetic and geometric progression (see Theorem 3.1 and 3.2). The results have always, as in the foregoing special case, coincided.

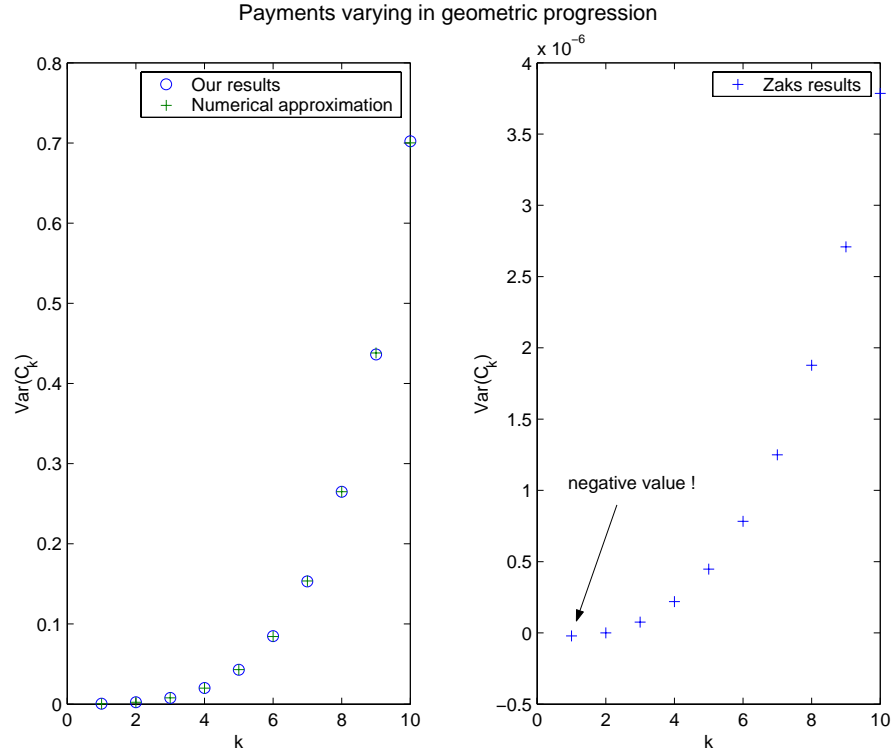


FIGURE 1. Comparison of the analytical (+) and numerical (o) results on variance of the final value of an annuity-due with payments varying in geometric progression with $p = 1$ and $q = 1 + u$. The right panel applies to Theorem 4.6 from Zaks (2001) and the left one to Corollary 3.5.

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